Curvature of Surfaces in $\mathbb{R}^3$

Contents

1 Introduction

In this essay, we will explore the concept of curvature, firstly for curves and then generalising for surfaces. We will introduce the notion of the Gauss Map and use it the define the Gaussian Curvature of a surface. Finally, we will build up to the *Theorema Egregium*, a fundamental theorem discovered by Gauss himself.

2 Recap

2.1 Curves and their Parametrisations

Recall from **Geometry and Motion** that curves can be viewed as *One-Dimensional* objects in $\mathbb{R}^n$. If $C \subseteq \mathbb{R}^n$ is a curve, then if

$$r(t) : I \longrightarrow \mathbb{R}^n$$

is a continuous vector function such that:

$$C = \{r(t) : t \in I\}$$

then we call the mapping r a *Parametrisation* of C.
Note that r is not unique!

One important thing to consider, is that a parametrisation of a curve fixes its orientation.

Definition 2.1. (Regular Parametrisation)
A parametrisation r of a curve C is *regular* if r' is defined and non-zero at all points along the curve.

Definition 2.2. (Arc Length/Natural Parametrisation)
A regular parametrisation r of a curve C is *natural* if $\|r'(s))\| = 1$

2.2 Curvature of a Curve

Definition 2.3. (Unit Tangent Vector)
Suppose that $r(t)$ is a natural parametrisation of C. Then, we define the *unit tangent vector* to be

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

Definition 2.4. Using the Unit Tangent Vector, we can intuitively define a notion of curvature by considering how quickly T changes as we move along the curve.
Hence we define the *curvature* κ of C at a point s as follows:

$$\kappa(s) = \left\| \frac{\mathrm{d}T}{\mathrm{d}s}(s) \right\|$$

Suppose that $\rho(t)$ is any regular parametrisation of C. Then we must normalise by $\|\rho'\|$. So we then obtain:

$$\kappa(t) = \frac{\|T'(t)\|}{\|\rho'(t)\|}$$

We can interpret the curvature geometrically by using the concept of an osculating circle. The osculating circle is the circle that best fits the curve at a given point. We can then define the curvature in a different way as: $\kappa = \frac{1}{R}$ where R is the radius of the osculating circle.

Example 2.1. Consider the right-handed helix parametrised by

$$r(t) = (\cos(t), \sin(t), t).$$

First we calculate the Unit Tangent Vector

$$r'(t) = (-\sin(t), \cos(t), 1)$$

$$\|r'(t)\| = \sqrt{2}$$

Hence:

$$T(t) = \frac{\sqrt{2}}{2}(-\sin(t), \cos(t), 1)$$

Now to calculate the curvature, we must compute $\|T'(t)\|$.

$$T'(t) = \frac{\sqrt{2}}{2}(-\cos(t), -\sin(t), 0)$$

$$\|T'(t)\| = \frac{\sqrt{2}}{2}$$

Thus, we obtain:

$$\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\sqrt{2}/2}{\sqrt{2}} = \frac{1}{2}$$

So the helix has a constant curvature of 1/2.

2.3 Surfaces and their Parametrisations

Similar to curves, surfaces in $\mathbb{R}^3$ can be viewed as *two-dimensional* objects in $\mathbb{R}^3$.
So we have a surface S in $\mathbb{R}^3$ which can be parametrised by a continuous vector function mapping a region $\Omega \subseteq \mathbb{R}^2$ onto $\mathbb{R}^3$.
More explicitly:

$$r : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

Where $S = \{r(u, v) : (u, v) \in \Omega\}$.

2.4 Unit Normal Vector

As $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ form a basis of the tangent plane to S, the normal vector to the surface must be orthogonal to both of these.

Hence, after normalising, we obtain the following formula:

Definition 2.5. Suppose that S is a surface parametrised by $r(u, v)$ with $r(u', v') = p \in S$.

Then the unit normal vector at p is given by:

$$N = \pm \frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}$$

Where the sign determines the orientation of the normal and with all the partial derivatives evaluated at (u', v').

2.5 Tangent Plane

An important concept that we will need later is that of the Tangent Plane. It can be defined in various ways, however, we will define it as follows:

Definition 2.6. (Tangent Plane)

Let $S \subseteq \mathbb{R}^3$ be a surface and $p \in S$. Then the tangent plane to S at p is:

$$T_p(S) = \{x \in \mathbb{R}^3 : \langle x, N(p) \rangle = 0\}$$

where $N(p)$ is the unit normal vector to the surface at p.

3 The Gauss Map

3.1 Defining the Gauss Map

Suppose that S is a regular, orientable surface in $\mathbb{R}^3$, then we can fix an orientation of S by *fixing* the unit normal vector N at a point of the surface. Note that, as N is a unit vector, we can identify the image of N as a subset of the unit sphere (which we refer to as S^2). So, defined in this way, we call $N : S \to S^2$ the Gauss Map. [1, p. 136]

As we require S to be regular and orientable, it follows that N is differentiable.

Definition 3.1. Suppose that S is a surface in $\mathbb{R}^3$ and $f : S \to \mathbb{R}^n$ is a differentiable vector function. Then we define the *differential* of f at a point $p \in S$ as follows: For any given direction $v \in T_pS$, we choose a curve $\alpha : (-\delta, \delta) \to S$ such that: $\alpha(0) = p$ and $\alpha'(0) = v$

We then denote the differential by df_p according to:

$$df_p(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \alpha)(t) = (f \circ \alpha)'(0)$$

[4, p. 46]

dN can then be viewed as a linear map from a tangent plane of the surface to a tangent plane of the unit sphere (for some given point $p \in S$). i.e. $dN_p : T_p(S) \to T_{N(p)}(S^2)$. Note, however, that $T_p(S)$ and $T_{N(p)}(S^2)$ are parallel (isomorphic), then we can just view dN_p as a linear map on $T_p(S)$ [3, p. 10].

As dN takes values from the tangent plane of a point, we can use it to observe how quickly a surface "*pulls aways*" from its tangent plane at a point. This concept will help us to construct an idea of a surface's curvature. i.e. the more 'curved' a surface is, the faster it pulls away from the tangent plane.

Definition 3.2. A linear map $A : V \to V$ is *self-adjoint* with respect to an inner product $\langle \cdot, \cdot \rangle$ if $\langle Av, w \rangle = \langle v, Aw \rangle$ for all $v, w \in V$ [1, p. 214]

An important property of the Gauss map is that its differential is self-adjoint.

Proposition 3.1. *The linear map $dN_p : T_p(S) \to T_p(S)$ is self-adjoint with respect to the standard inner product restricted on $T_p(S)$.*

Proof. We use the proof given in [1, p. 140-141] with some slight variations.

Due to the linearity of dN_p it is sufficient to prove $\langle dN_p(e_1), e_2 \rangle = \langle e_1, dN_p(e_2) \rangle$ for some basis (e_1, e_2) of $T_p(S)$.

Let $r(u, v)$ be a parametrisation of S with $r(u', v') = p \in S$. We then have $\left(\frac{\partial r}{\partial u}(u', v'), \frac{\partial r}{\partial v}(u', v')\right)$ as a basis for $T_p(S)$.

Now, consider a curve α in S with $\alpha(t) = r(u(t), v(t))$ and $\alpha(0) = p$. We can now use the definition of a differential and the chain rule as follows:

$$\begin{aligned} dN_p(\alpha'(0)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (N(\alpha(t))) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (N(u(t), v(t))) \\ &= \frac{\partial N}{\partial u} u'(0) + \frac{\partial N}{\partial v} v'(0) \end{aligned}$$

If $\alpha'(0) = \frac{\partial r}{\partial u}$ then it follows that $u'(0) = 1$ and $v'(0) = 0$, and vice versa when swapping u and v. From this, we can see that:

$$dN_p \left(\frac{\partial r}{\partial u} \right) = \frac{\partial N}{\partial u}$$

$$dN_p \left(\frac{\partial r}{\partial v} \right) = \frac{\partial N}{\partial v}$$

We now change notation for ease of reading, r_u will now denote $\frac{\partial r}{\partial u}$, etc.

All we must prove now is the equality:

$$\langle N_u, r_v \rangle = \langle r_u, N_v \rangle$$

As N, r_u, r_v are all orthogonal to each other, we can take derivatives of $\langle N, r_u \rangle = 0$ with respect to v and $\langle N, r_v \rangle = 0$ with respect to u. [3, p. 19]

This gives:

$$\langle N_v, r_u \rangle + \langle N, r_{uv} \rangle = 0$$
$$\langle N_u, r_v \rangle + \langle N, r_{vu} \rangle = 0$$

As r is a sufficiently *nice* function, we note that the second order partial derivatives commute for this function. Hence, we obtain the required result

$$\langle N_u, r_v \rangle = \langle r_u, N_v \rangle$$

□

3.2 The 2nd Fundamental Form

We can also show the following:

Proposition 3.2. *If a linear map $A : V \to V$ is self-adjoint, then the matrix of A with respect to an orthonormal basis is symmetric.*

Proof. Let (a_{ij}) be the matrix of A with respect to some orthonormal basis (e_1, e_2).

We then see using the fact that A is self-adjoint and the inner product $\langle \cdot, \cdot \rangle$ is symmetric.

$$a_{ij} = \langle Ae_i, e_j \rangle = \langle e_i, Ae_j \rangle = \langle Ae_j, e_i \rangle = a_{ji}$$

[1, p. 214] □

As dN_p is self-adjoint, we can associate it to a quadratic form in the tangent plane defined as follows:

Definition 3.3. (Second Fundamental Form) The *quadratic form* Π_p, defined on $T_p(S)$ by:

$$\Pi_p(\mathbf{v}) = -\langle dN_p(\mathbf{v}), \mathbf{v} \rangle$$

is called the Second Fundamental Form of S at p. [1, p. 141]

4 Defining Curvature

4.1 Normal and Principal Curvatures

Definition 4.1. Let $C \subseteq S$ be a regular curve passing through a point $p \in S$. Let k be the curvature of C at p, n the normal vector to C and N the normal vector to S. Then the normal curvature of C at p is:

$$k_n = k\cos(\theta) = k\langle n, N\rangle$$

i.e. θ is the angle between n and N.

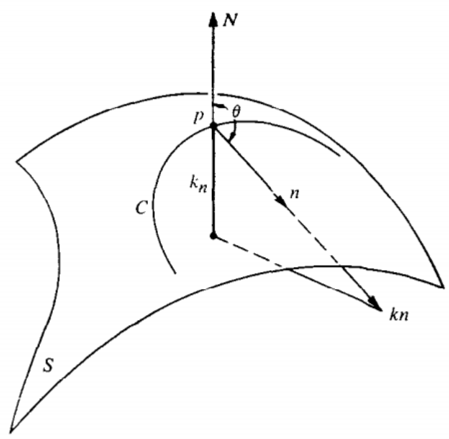

Figure 1: A visualisation of the normal curvature [1, p. 141]

Remark: The normal curvature does not change with the orientation of C. However, it changes sign if we reverse the orientation of S.

We can now utilise the 2nd Fundamental Form to easily compute the normal curvature.

In order to do this, we consider a regular differentiable curve parametrised by $r : (-\epsilon, \epsilon) \longrightarrow S$ such that $r(0) = p$. Then, we have: $\Pi_p(r'(0)) = k_n(p)$. [1, p. 142]

We now quote the following proposition without proof.

Proposition 4.1. *All curves lying in S that have the same tangent vector at a point p, also have the same normal curvature at p.*

Hence, using this result, we can calculate the normal curvature for any given $v \in T_p(S)$ by considering any curve with its tangent at p equal to v.

Definition 4.2. (Principal Curvatures)

We define the principal curvatures k_1, k_2 as the maximum and minimum (respectively) normal curvatures at p.

Using the 2nd Fundamental Form, we see that the principal curvatures represent the maximum and minimum values of Π_p. So, using this fact, we know that these values will correspond to the eigenvalues of dN_p.

4.2 Gaussian Curvature

Now we can finally define the notion of Gaussian Curvature.

Definition 4.3. (Gaussian Curvature)
Let $p \in S$ and dN_p be the differential of the Gauss map at p. Then, the *Gaussian Curvature* K at the point p is equal to $\det(dN_p)$.

We know that the principle curvatures correspond to the eigenvalues of dN_p, so we can represent the matrix of dN_p in a diagonal/upper triangular form by using the (generalised) eigenvectors as a basis. As the determinant does not depend on the basis, we see that the determinant is simply equal to the product of the diagonal entries. This is precisely the product of the eigenvalues, which in this form, is the principle curvatures!

Hence, we obtain an equivalent definition in terms of the principal curvatures:

$$K = k_1 k_2$$

[1, p. 146]

4.3 Examples

Now we consider some examples of surfaces and their curvatures.

Example 4.1. Let S be the unit cylinder, parallel to the z-axis. Let $p = (x, y, z)$ be a point on the cylinder.

As the z-coordinate has no effect on the unit normal vector, we see that: $N(x, y, z) = (x, y, 0)$.

The linear map dN_p is then given by the matrix:

$$dN_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has a row of zeroes, so its determinant is equal to 0. Hence, the curvature of the cylinder is 0 at every point.

This is quite a surprising example, as we think of the cylinder as being curved. However, we will see soon, that it fits our definition and it makes sense that it has zero curvature.

Example 4.2. Let S^2 be the unit sphere and $p = (x, y, z)$ be a point on the sphere.

In this example, the Gauss map is the same as the identity map. i.e. $N(x, y, z) = (x, y, z)$.

So we obtain:

$$dN_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which has determinant 1. So, every point on the sphere has constant curvature $K = 1$

Example 4.3. We now consider our surface S to be the x-y plane.

Here, the normal vector is constant: $N(x, y, z) = (0, 0, 1)$ and so the differential is the zero map. Clearly, the determinant is 0 and we conclude that every point on the plane has constant curvature $K = 0$.

5 Theorema Egregium

We can now state the key theorem concerning curvature of surfaces. This theorem was first proved by Gauss who called it *Theorema Egregium* or *Remarkable Theorem.*

Definition 5.1. Let U be a neighbourhood of $p \in S$. Then, a map $\phi : U \to S'$ is a *local isometry* at p if there exists a neighbourhood $V \subseteq S'$ of $\phi(p)$ such that $\phi : U \to V$ is an isometry.

We then say that two surfaces S, S' are *locally isomorphic* if there exists a local isometry into S' at every point in S. [1][p. 219]

Theorem 5.1. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged[2, p. 20].*

In other words: if there exists a local isometry between two surfaces, then they both have the same curvature at corresponding points.

The proof of this theorem is long and requires a lot of algebraic manipulation with use of the Weingarten equations, so it will be omitted.

However, we can now observe some consequences of this theorem.

Firstly, we consider the example of the cylinder and the plane. We can interpret the Theorema Egregium as saying that Gaussian Curvature is "*bending invariant*" [5, p. 133]. So, in our two examples, we can see that we can roll the plane into the cylinder, so it makes sense that they both have the same curvature.

Secondly, we consider the sphere and the plane. These have different curvatures, hence, by the theorem we can prove that there does not exist a local isometry between them. This has applications in cartography as it shows that it is impossible to map the Earth without distorting either the distance or the angles.

References

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